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CATEGORICAL PRODUCT OF TWO S-VALUED GRAPHS

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Abstract

Motivated by the study of products in crisp graph theory and the notion of S-valued graphs, in this paper, we study the concept of categorical product of two S-valued graphs.

1. Introduction

Graphs are, of course, basic combinatorial structures. Products of structures are a fundamental construction in mathematics, for which theorems abound in set theory, category theory, and universal algebra. One can expect many of the nice properties of products to be a result of a role played in some Category-theoretic construct. However, the graph product such as, [6] Cartesian product, Categorical product, Strong product and the lexicographic product do not arrive in that way. Thus, it is not surprising that good things happen when we take products of graphs; many unique and new ideas emerge.

Key Words : *Graph operations, Product of Graphs, Semiring, S-valued graphs.*
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Algebraic graph theory can be viewed as an extension of graph theory in which algebraic methods are applied to problems about graphs [1]. Recently in [4], Chandramouleeswaran et.al. introduced the concept of semiring valued graphs, called S-valued graphs. In [2] the authors have studied the regularity of S-valued graphs. In [5], the authors have discussed the concept of degree regular S-valued graphs. This motivated us to study, the notion of categorical product of two S-valued graphs and their regularity properties.

2. Preliminaries

In this section, we recall the basic definitions that are needed in the sequel.

Definition 2.1. [6] : The Categorical product of two graphs $G = (V(G); E(G))$ and $H = (V(H), E(H))$ is the graph denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus $V(G \times H) = \{(g, h) | g \in V(G)$ and $h \in V(H)\}$, $E(G \times H) = \{(g, h)(g', h') | gg' \in E(G)$ and $hh' \in E(H)\}$.

Definition 2.2 [3] : A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

(1) $(S, +, \cdot)$ is a monoid.

(2) $(S, \cdot)$ is a semigroup.

(3) For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

(4) $0 \cdot x = x \cdot 0 = 0 \ \forall x \in S$.

The element 0 in S is called the additive identity of the Semiring S.

Definition 2.3 [3] : Let $(S, +, \cdot)$ be a semiring. $\preceq$ is said to be a canonical pre- order if for $a, b \in S$, $a \preceq b$ if and only if there exists $c \in S$ such that $a + c = b$.

Definition 2.4 [4] : Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring valued graph (or a S-valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\} & \text{if} \quad \sigma(x) \preceq \sigma(y) \quad \text{(or)} \quad \sigma(y) \preceq \sigma(x) \\ 0 & \text{otherwise} \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. we call σ, a S-vertex set and ψ an S-edge set of S-valued graph G^S.

Definition 2.5 [**4**] : Let $G^S = (V, E, \sigma, \psi)$ be the S-valued graph corresponding to a given crisp graph $G = (V, E)$. A S-valued graph $H^S = (P.L, \tau, \gamma)$ is called a S-subgraph of G^S if $H = (P, L)$ is a subgraph of G with $P \subset V, L \subset E$, $\tau \subset \sigma$ and $\gamma \subset \psi$. That is $\tau \subset \sigma \Rightarrow \tau(x) \preceq \sigma(x), x \in P$ and $\gamma \subset \psi \Rightarrow \gamma(x, y) \preceq \psi(x, y)$, $(x, y) \in L \subset P \times P$.

Definition 2.6 [**4**] : Let $G^S = (V, E, \sigma, \psi)$ be a S-valued graph and $H^S = (P, L, \tau, \gamma)$ be its S-subgraph. H^S is called a S-subgraph of G^S induced by P if $P \subset V$, $L \subset E$, $\tau(x) = \sigma(x)$, for every $x \in P$ and $\gamma(x, y) = \psi(x, y)$ for every $(x, y) \in L$.

Definition 2.7 [**4**] : The open neighbourhood of v_i in G^S is defined as

$$N_S(v_i) = \{(v_j, \sigma(v_j)) | (v_i, v_j) \in E \ \psi(v_i, v_j) \in S\}.$$

The closed neighbourhood of v_i in G^S is defined as $N_S[v_i] = N_S(v_i) \cup \{(v_i, \sigma(v_i))\},$.

Definition 2.8 [**5**] : The degree of a vertex v_i of the S-valued graph G^S is defined as $deg_S(v_i) = \left(\sum_{v_j \in N_S(v_i)} \psi(v_i, v_j), l \right)$ where l is the number of edges incident with v_i.

Definition 2.9 [**4**] : A S-valued graph G^S is said to be

(1) vertex regular if $\sigma(v) = a \ \forall \ \ v \in V$ and for some $a \in S$

(2) Edge regular if $\psi(u, v) = a \ \forall \ (u, v) \in E$ and for some $a \in S$.

(3) S-regular if it is both vertex as well as edge regular.

Definition 2.10 [**5**] : A S-valued graph G^S is said to be degree regular S-valued graph (d_S-regular graph) if $deg_S(v) = (a, n) \ \forall \ v \in V$ and some $a \in S$ and $n \in Z_+$.

Definition 2.11 [**2**] : A graph G^S is said to be (a, k) regular if the underlying crisp graph G is k-regular and $\sigma(v) = a, \ \forall \ v \in V$.

3. Categorical Product of Two S-valued Graphs

In this section, we introduce the notion of categorical product of two S-valued graphs, illustrate with some examples, and prove simple properties.

Definition 3.1 : Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i | 1 \leq i \leq p_1\}$, $E_1 \subseteq V_1 \times V_1$ and $G_2^S = (V_1, E_2, \sigma_2, \psi_2)$ where $V_2 = \{u_j | 1 \leq j \leq p_2\}$, $E_2 \subseteq V_2 \times V_2$ be two given S-valued graphs.

The Categorical product of two S-valued graphs G_1^S and G_2^S is defined by

$$G_\times^S = G_1^S \times G_2^S = (V, E, \sigma, \psi),$$

where $V = V_1 \times V_2 = \{w_{ij} = (v_i, u_j) | v_i \in V_1, u_j \in V_2\}$; $1 \leq i \leq p_1$; $1 \leq j \leq p_2$ the two vertices $w_{ij} = (v_i, u_j)$, $w_{kl} = (v_k, u_l)$ are adjacent if $v_i v_k \in E_1$ and $u_j u_l \in E_2$. Then $E = \{e_{ij}^{kl} = (w_{ij}, w_{kl}) | e_i^k = v_i v_k \in E_1 \text{ and } e_j^l = u_j u_l \in E_2\}$.

Define the S-valued functions, $\sigma : V \to S$ by $\sigma(v_i, u_j) = \min\{\sigma_1(v_i), \sigma_2(u_j)\}$ and $\psi : E \to S$ by $\psi(w_{ij}) = \min\{\psi_1(e_i^k), \psi_2(e_j^l)\}$.

Example 3.2 : Consider the semiring $S = (\{0, a, b, c\}, +, \cdot)$ with the binary operations '+' and '$\cdot$ defined by the following Cayley tables.

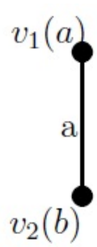
$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

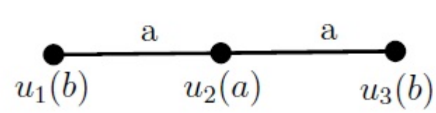
$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

In S we define a canonical pre-order $\preceq$ by

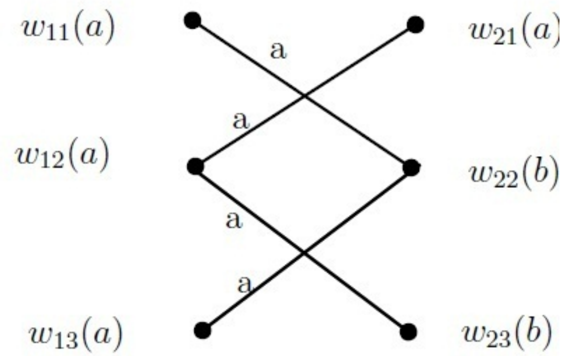
$$0 \preceq 0,\ \ 0 \preceq a, 0 \preceq b, 0 \preceq c, a \preceq a, b \preceq b, c \preceq c, a \preceq b, a \preceq c, b \preceq c.$$

Consider the two S-valued graphs G_1^S and G_2^S:





Then the categorical product $G_\times^S = G_1^S \times G_2^S = (V, E, \sigma, \psi)$ is



Here, $V = \{w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}\}$ and $E = \{e_{11}^{22}, e_{12}^{21}, e_{12}^{23}, e_{13}^{22}\}$.
The S-vertex set of $G_1^S \times G_2^S = \{a, b\}$ and the S-edge set of $G_1^S \times G_2^S = \{a\}$.

Theorem 3.3 : The Categorical product of two S-regular graph is again a S-regular graph.

Proof : Let $G_1^S = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be two S-regular graphs. That is, G_1^S and G_2^S is both vertex regular as well as edge regular.

Claim : $G_\times^S = (V, E, \sigma, \psi)$ is S-regular. That is to prove that, $\sigma(w_{ij})$ is equal for all $w_{ij} \in V$ and $\psi(e_{ij}^{kl})$ is equal for all $e_{ij}^{kl} \in E$.
Now by definition

$$\sigma(w_{ij}) = \min\{\sigma_1(v_i), \sigma(u_j)\} = \begin{cases} \sigma_1(v_i) & \text{if} \quad \sigma_1(v_i) \preceq \sigma_2(u_j) \\ \\ \sigma_2(u_j) & \text{if} \quad \sigma_2(u_j) \preceq \sigma_1(v_i) \end{cases}$$

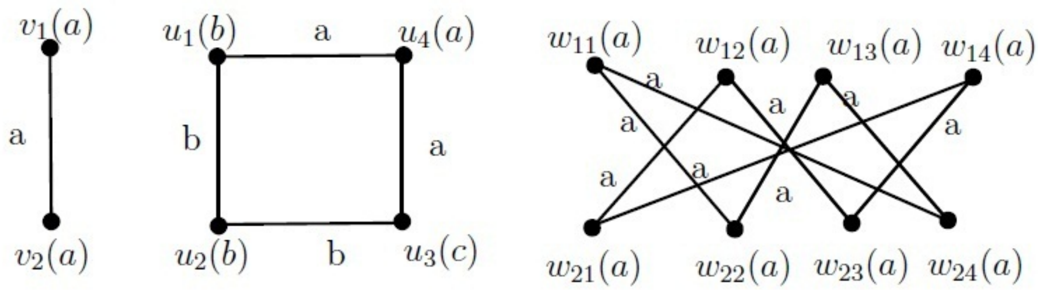
Then in both the cases $\sigma(w_{ij})$ is equal for all $w_{ij} \in V, 1 \le i \le p_1$, $1 \le j \le p_2$.
This implies that $G_x^S = G_1^S \times G_2^S$ is vertex regular. Further,

$$\begin{aligned} \psi(e_{ij}^{kl}) &= \min\{\psi_1(e_i^k), \psi_2(e_j^l)\} \\ &= \min\{\sigma_1(v_i), \sigma_2(u_j)\} \quad (\because \;\; G_1^S \;\; and \;\; G_2^S \;\; are \;\; S-regular) \\ &= \begin{cases} \sigma_1(v_i) & \text{if} \quad \sigma_1(v_i) \preceq \sigma_2(u_j) \\ \\ \sigma_2(u_j) & \text{if} \quad \sigma_2(u_j) \preceq \sigma_1 v_i) \end{cases} \end{aligned}$$

Thus $\psi(e_{ij}^{kl}) = \sigma_1(v_i)$ or $\psi(e_{ij}^{kl}) = \sigma_2(u_j)$ for all eges $e_{ij}^{kl} \in E$.
This implies that, $G_\times^S = G_1^S \times G_2^S$ is edge regular.
Thus the Categorical product, $G_\times^S = G_1^S \times G_2^S$ is a S-regular graph.

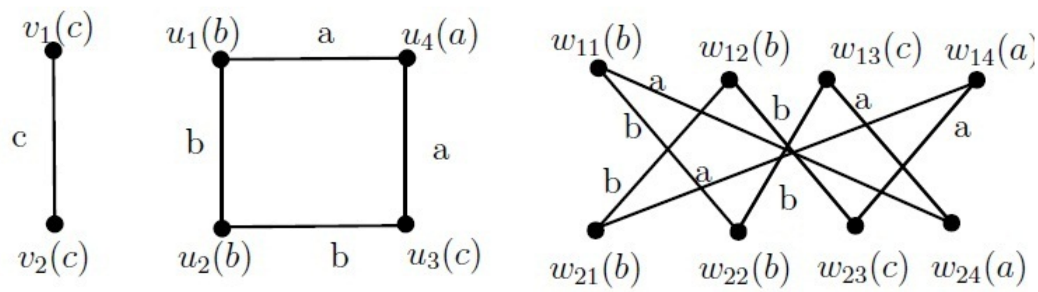
The converse of the above theorem need not be true in general, as seen from the following example.

Example 3.4 : Consider the semiring $(S = \{0, a, b, c\}, +, \cdot)$ as in the example 3.2. Consider the S-valued graphs G_1^S and G_2^S and its categorical product $G_\times^S$:



Clearly $G_\times^S = G_1^S \times G_2^S$ and G_1^S are S-regular while G_2^S is not.
However, the following example gives that the product G_x^S is not S-regular eventhough one of the factors G_1^S or G_2^S is S-regular.

Example 3.5 : Consider the semiring $S = \{a, b, c\}, +, \cdot$ as in the example 3.2. Consider the S-valued graphs G_1^S and G_2^S and its categorical product $G_\times^S$:



$G_\times^S = G_1^S \times G_2^S$ is not Sregular, eventhough G_1^S is S-regular while G_2^S is not for $b \preceq c$ in S.

The above example leads to the following theorem.

Theorem 3.6 : The Categorical product of two S-valued graphs is S-regular if the S-value corresponding to the S-regular graph is minimum among the S-values.

Proof : Let G_1^S and G_2^S be two S-valued graphs such that G_1^S is S-regular with the S-value is minimum among the S-values.

That is,$\sigma_1(v_i) = a$ for all $v_i \in V_1$ and for some $a \in S$, which is minimum in the semiring S.

Claim : $G^S_\times = G^S_1 \times G^S_2$ is S-regular with the S-value $a \in S$.

Now, For any $w_{ij} \in V$,

$$\begin{aligned}\sigma(w_{ij}) &= \min\{\sigma_1(v_i), \sigma_2(u_j)\} = \min\{a, \sigma_2(u_j)\} \\ &= a \quad (\because \quad a \preceq \sigma_2(u_j) \ \ \forall \ \ u_j \in V_2).\end{aligned}$$

Thus $G^S_\times = G^S_1 \times G^S_2$ is a vertex regular.

Since every vertex regular S-valued graph is edge regular, $G^S_\times$ is edge regular.

Then, $G^S_\times$ is S-regular with the S-value a, which is minimum among the S-values. This proves that $G^S_\times$ is S-regular with S-value $\min\{\sigma_1(v_i), \sigma_2(u_j), 1 \le i \le p_1, 1 \le j \le p_2\}$.

Theorem 3.7 : The Categorical product of two edge regular S-valued graphs is an edge regular S-valued graph.

Proof : Let$G^S_1 = (V_1, E_1, \sigma_1, \psi_1)$ and $G^S_2 = (V_2, E_2, \sigma_2, \psi_2)$ be two edge regular S-valued graphs.

Then $\forall \ \ e^k_i \in E_1$ and $\forall \ \ e^l_j \in E_2$, the values $\psi_1(e^k_i)$ and $\psi_2(e^l_j)$ are all equal.

Claim : $G^S_\times = G^S_1 \times G^S_2$ is an edge regular S-valued graph.

That is to prove that $\psi(e^{kl}_{ij})$ is equal for every $e^{kl}_{ij} \in E$. By definition

$$\psi(e^{kl}_{ij}) = \min\{\psi_1(e^k_i), \psi_2(e^l_j)\} = \begin{cases} \psi_1(e^k_i) \quad if \quad \psi_1(e^k_i) \preceq \psi_2(e^l_j) \\ \\ \psi_2(e^l_j) \quad if \quad \psi_2(e^l_j) \preceq \psi_1(e^k_i) \end{cases}$$

This implies that, $\psi(e^{kl}_{ij})$ is equal for every edges $e^{kl}_{ij} \in E$.

Thus the categorical product of G^S_1 and G^S_2, $G^S_\times = G^S_1 \times G^S_2$ is again an edge regular S-valued graph.

Theorem 3.8 : The Categorical product of two degree regular S-valued graphs (d_S-regular) is again a degree regular S-valued graph.

Proof : Let G^S_1 and G^S_2 be two degree regular S-valued graphs.

Then $deg_S(v_i) = \left(\sum\limits_{v_k \in N_S(v_i)} \psi_1(v_i v_k), m \right) = (a, m)$ for some $a \in S$ and $\forall \ \ v_i \in V_1$.

$deg_S(u_j) = \left(\sum\limits_{u_l \in N_S(u_j)} \psi_2(u_j u_l), n \right) = (b, n)$ for some $b \in S$ and $\forall \ \ u_j \in E_2$.

Claim : $G^S_\times = G^S_1 \times G^S_2$ is degree regular S-valued graph. That is to prove that for all vertices $w_{ij} \in V, deg_S(w_{ij}) = \left(\sum_{w_{kl} \in N_S(w_{ij})} \psi(w_{ij}w_{kl}), r \right)$ is equal where r is the number of incident edges of $w_{ij} = (v_i, u_j)$ in $G^S_\times$.

In crisp graph, the number of incident edges of $w_{ij} = (v_i, u_j)$ is equal to the product of number of incident edges of v_i and the number of incident edges of u_j.

That is, The no. of incident edges of $w_{ij} =$

The no. of incident edges of $v_i \times$ The no. of incident edges of u_j.

This implies that, $r = m \times n = mn$.

Then, for any vertices $w_{ij} \in V,\ 1 \leq i \leq p_1; 1 \leq j \leq p_2$.

$$\begin{aligned} deg_S(w_{ij}) &= \left(\sum_{w_{kl} \in N_S(w_{ij})} \psi(e^{kl}_{ij}), mn \right) \\ &= \left(\sum_{w_{kl} \in N_S(w_{ij})} \min\{\psi_1(e^k_i), \psi_2(e^l_j)\}, mn \right) \\ &= \left(\sum_1^{mn} \min\left\{ \sum_{k=1}^{m} \psi_1(e^k_i), \sum_{l=1}^{n} \psi_2(e^l_j) \right\}, mn \right). \end{aligned}$$

Since G^S_1 is d_S-regular, $\sum_{k=1}^{m} \psi_1(e^k_1)$ is eqaul for all edges $e^k_i \in E_1$ and G^S_2 is d_S-regular, $\sum_{l=1}^{n} \psi_2(e^l_j)$ is equal for all edges $e^l_j \in E_2$.

Thus for all vertices $w_{ij} \in V$ of $G^S_\times$,

$$deg_S(w_{ij}) = \left(\sum_1^{mn} \min\left\{ \sum_{k=1}^{m} \psi_1(e^k_i)\psi_2(e^l_j) \right\}, \min \right)$$

is equal.

Hence, every vertices of $G^S_\times = G^S_1 \times G^S_2$ have the same degree.

This implies that, the Categorical product $G^S_\times = G^S_1 \times G^S_2$ is a degree regular S-valued graph.

Proposition 3.9 : Let $(S, +, \cdot)$ be a semiring and $a, b \in S$. If G^S_1 is (a, m)-regular graph and G^S_2 is (b, n)-regular graph then the Categorical product $G^S_\times = G^S_1 \times G^S_2$ is either (a, mn)-regular or (b, mn)-regular graph, depending on $a \preceq b$ or $b \preceq a$ respectively.

Proof : Let $G^S_1 = (V_1, E_1, \sigma_1, \psi_1)$ be a (a, m)-regular graph for some $a \in S$ and $m \in Z_+$. Then $\sigma_1(v_i) = a$ for all $v_i \in V_1$ and no. of incident edges of v_i is m. (1)

Let $G_2^S = (V_2, E_2, \sigma_2, \psi_2)$ be a (b, n)-regular graph for some $b \in S$ and $n \in Z_+$.
Then $\sigma_2(u_j) = b$ for all $u_j \in V_2$ and no. of incident edges of u_j is n. (2)
Claim : $G_x^S = G_1^S \times G_2^S$ is either (a, mn)-regular or (b, mn)-regular graph.
Then we have to prove that $\sigma(w_{ij}) = a$ or $\sigma(w_{ij}) = b$ for some $a, b \in S$ and the number of incident edges of w_{ij} is $k \in Z_+$ for all vertices $w_{ij} \in V, 1 \leq i \leq p_1; 1 \leq j \leq p_2$.
By Definition

$$\begin{aligned} \sigma(w_{ij}) &= \min\{\sigma_1(v_i), \sigma_2(_j)\} \\ &= \min\{a, b\} \quad \text{(By (1) and (2))} \\ &= \begin{cases} a \quad if \quad a \preceq b \\ \\ b \quad if \quad b \preceq a \end{cases} \end{aligned}$$

Thus for all vertices $w_{ij} \in V$, $\sigma(w_{ij}) = b$, $1 \leq i \leq p_1$; $1 \leq j \leq p_2$.
This implies that, in both the cases, $G_\times^S = G_1^S \times G_2^S$ is vertex regular.
In $G_\times^S = G_1^S \times G_2^S$, the number of incident edges of any vertices w_{ij} is equal to the product of number of incident edges of v_i and the number of incident edges of u_j.
Then from equation (1) and (2), we have
The no. of $\cdot$ incident edges of $w_{ij} = m \times n = mn = k$ (say).
Thus the categorical product $G_\times^S = G_1^S \times G_2^S$ is vertex regular as well as all the vertices have the same number of incident edges.
Hence, $G_\times^S = G_1^S \times G_2^S$ is either (a, k)-regular or (b, k)-regular graph.

4. Conclusion

Motivated by the study of S-valued graphs in [4], [3] and [5], we studied the regularity and degree regularity conditions on the categorical product of two S-valued graphs. In future, we have proposed to study the notions of minimal and maximal degree and their properties on $G_\times^S$.

References

[1] Chris Godsil and Gordon Royle. Algebraic Graph Theory, Springer, (2001).

[2] Jeyalakshmi S., Rajkumar M. and Chandramouleeswaran M., Regularity on S-valued graphs, Global J. of Pure and Applied Maths., 2(5) (2015), 2971-2978.

[3] Jonathan Golan, Semirings and Their Applications, Kluwer Academic Publishers, London.

[4] Rajkumar M., Jeyalakshmi S. and Chandramouleeswaran M., Semiring valued graphs, International Journal of Math. Sci. and Engg. Appls., 9(3) (2015), 141-152.

[5] Rajkumar M. and Chandramouleeswaran M., Degree regular S-valued graphs, Mathematical Sciences International Journal of Math. Sci. and Engg. Appls., 4(2) (2015).

[6] Richard Hammack, Wilfried Imrich and Sandy Klavzar, Handbook of Product Graphs, University of Ljubljana and University of Maribor, Slovenia.